Parabolic Anderson model with a fractional Gaussian noise that is rough in time

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The main topic

Consider the Parabolic Anderson model

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \dot{W}(t,x)u(t,x) \\ u(0,x) = u_0(x) \end{cases}$$

where $W(t,x)$ ($(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$) is an $(1+d)$-dimensional fractional Brownian sheet with the Hurst parameter $(H_0, \cdots, H_d)$ ($0 < H_0, \cdots, H_d < 1$) define as the mean-zero Gaussian field with the covariance function

$$\mathrm{Cov}\left(W(s,x), W(t,y)\right) = R_{H_0}(s,t)\prod_{j=1}^{d} R_{H_j}(x_j, y_j)$$

for $x = (x_1, \cdots, x_d)$, $y = (y_1, \cdots, y_d)$ and

$$R_H(u,v) = \frac{1}{2}\left\{|u|^{2H} + |v|^{2H} - |u-v|^{2H}\right\} \quad u.v \in \mathbb{R}$$

The main topic

To determine the covarince for the generalized Gaussian field

$$\dot{W}(t,x) = \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}(t, x_1, \cdots, x_d)$$

we conduct the formal computation

$$\begin{aligned}
&\mathrm{Cov}\left(\dot{W}(s,x), \dot{W}(t,y)\right) \\
&= \frac{\partial^{2(d+1)}}{(\partial s \partial t)(\partial x_1 \partial y_1) \cdots (\partial x_d \partial y_d)} \mathrm{Cov}\left(W(s,x), W(t,y)\right) \\
&= \frac{\partial^2 R_{H_0}(s,t)}{\partial s \partial t} \prod_{j=1}^{d} \frac{\partial^2 R_{H_j}(x_j, y_j)}{\partial x_j \partial y_j}
\end{aligned}$$

The main topic

When $\mathrm{H} > 1/2$,

$$\frac{\partial^2 \mathrm{R_H}(\mathrm{u},\mathrm{v})}{\partial \mathrm{u} \partial \mathrm{v}} = -\frac{1}{2}\frac{\partial^2}{\partial \mathrm{u}\partial \mathrm{v}}\{|\mathrm{u}-\mathrm{v}|^{2\mathrm{H}}\} \\ = \mathrm{H}(2\mathrm{H}-1)|\mathrm{u}-\mathrm{v}|^{-(2-2\mathrm{H}_0)} \tag{1}$$

When $\mathrm{H} = 1/2$

$$\frac{\partial^2 \mathrm{R}_{1/2}(\mathrm{u},\mathrm{v})}{\partial \mathrm{u} \partial \mathrm{v}} = -\frac{1}{2}\frac{\partial^2}{\partial \mathrm{u}\partial \mathrm{v}}\{|\mathrm{u}-\mathrm{v}|\} = \delta_0(\mathrm{u}-\mathrm{v}) \tag{2}$$

The identification (1) can not extended to the setting $\mathrm{H} < 1/2$, as the function $-|\cdot|^{-(2-2\mathrm{H})}$ is disqualified as the covarince function for it is not non-negative definite.

The main topic

On the other hand, (1) and (2) can be unified as

$$\frac{\partial^2 R_H(u,v)}{\partial u \partial v} = C_H \int_{\mathbb{R}} e^{i\lambda(u-v)} |\lambda|^{1-2H} d\lambda \tag{3}$$

with the suitable constant $C_H > 0$. We remark that (3) leads to the requested definite non-negativity.

We extend the identity (3) to the case $H < 1/2$. Notice that the covariance function given in (3) is not point-wisely defined when $H < 1/2$, nor is it non-negative in any proper sense.

The main topic

In summary, the Gaussian noise is a generalized mean-zero Gaussian field with the covariance function

$$\mathrm{Cov}\left(\dot{W}(s,x),\dot{W}(t,y)\right)=\gamma_0(s-t)\gamma(x-y)$$

with the time and space covariance functions given as

$$\gamma_0(s-t)=\int_{\mathbb{R}} e^{i\lambda(s-t)}\mu_0(d\lambda) \text{ and } \gamma(x-y)=\int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\mu(d\xi)$$

and the time and space spectral measures give as

$$\mu_0(d\lambda)=C_0|\lambda|^{1-2H_0}d\lambda \text{ and } \mu(d\xi)=C_1\left(\prod_{j=1}^d |\xi_j|^{1-2H_j}\right)d\xi$$

The main topic

To see why $\gamma_0(\cdot)$ is sign-switching as $H_0 < 1/2$, we make the following formal computation:

$$\int_{\mathbb{R}} \gamma_0(u)du = C_0 \int_{\mathbb{R}} |\lambda|^{1-2H_0} \left[\int_{\mathbb{R}} e^{i\lambda u} du \right] d\lambda$$
$$= C_0 \int_{\mathbb{R}} |\lambda|^{1-2H_0} \delta_0(\lambda) d\lambda = |0|^{1-2H_0} = 0$$

The parabolic Anderson equation is interpreted as

$$u(t,x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) u(s,y) W^H(dsdx)$$

where $p_t(x)$ is the density of the Brownian semi-group and the stochastic integral is in the sense of Skorokhod.

The main topic

The setting of non-rough noise (i.e., $H_0, \cdots, H_d \geq 1/2$) has been well-understood. In this case, the system has a unique solution under the Dalang's condition

$$d - \sum_{j=1}^{d} H_j < 1$$

The current interest is in the case when some of $H_0, \cdots, H_d$ are less than $1/2$ (i.e., the noise $\dot{W}^H(t, x)$ is rough). In the $(1+1)$-dimension with $H_0 > 1/2$ and $1/4 < H_1 < 1/2$, the equation is solved by Hu-Huang-Nualart-Tindel (preprint), Chen-Hu-Nualart-Tindel (EJP, accepted) and Huang-Lê-Nualart (preprint).

Our theorem below shows in particular that the condition "$H_1 > 1/4$" is too restrictive as $H_0 > 1/2$ and the correct assumption should be $H_0 + H_1 > 3/4$.

Result in $H_0 \geq 1/2$

Set $J_* = \{1 \leq j \leq d;\ H_j < 1/2\}$,
$J^* = \{1 \leq j \leq d;\ H_j \geq 1/2\}$, $d_* = \#\{J_*\}$, $d^* = \#\{J^*\}$,

$$H_* = \sum_{j \in J_*} H_j, \quad H^* = \sum_{j \in J^*} H_j, \quad H = H_* + H^* = \sum_{j=1}^{d} H_j$$

Theorem (ALHP(to appear))

Let $H_0 \geq 1/2$. *Under the assumption*

$$\begin{cases} d - H < 1 & \textit{(Dalang's condition)} \\ 4(1 - H_0) + 2(d - H) + (d_* - 2H_*) < 4 \end{cases}$$

the parabolic Anderson equation admits a unique solution $u(t, x)$.

Main theorem

The major topic is this talk the setting when the Gaussian noise is rough in time, i.e., $H_0 < 1/2$.

Theorem

Assume $H_0 < 1/2$. Under the assumption

$$4(1 - 2H_0) + 2(d - H) + (d_* - 2H_*) < 2$$

the parabolic Anderson equation has a unique solution.

Summary in $(1+1)$-dimension

To see what we get in the above theorems, we consider the special case when case $d = 1$. So $\dot{W}$ is a $(1+1)$-fractional noise with Hurst parameter (H_0, H). According to our theorems, the system is solvable under any one of the following assumptions:

(1). $H_0, H \geq 1/2$;

(2). $H_0 \geq 1/2$, $H < 1/2$ and $H_0 + H > 3/4$;

(3). $H_0 < 1/2$, $H \geq 1/2$ and $4H_0 + H > 2$;

(4). $H_0 < 1/2$, $H < 1/2$ and $2H_0 + H > 5/4$.

Ito-Wiener chaos expansion

For simplicity we assume $u_0(x) = 1$ in the following discussion. By iterating the mild equation

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) u(s,y) W^H(dsdx)$$

we obtain the orthogonal decomposition

$$u(t,x) = 1 + \sum_{n=1}^{n} I_n\big(f_n(\cdot,t,x)\big)$$

known as Ito-Wiener chaos expansion. By orthogonality, the solvability of the mild equation is equivalent to

$$\mathbb{E}\, u^2(t,x) = 1 + \sum_{n=1}^{n} n! \| I_n\big(f_n(\cdot,t,x)\big) \|^2 < \infty$$

Relevance to Brownian Hamiltonian

It can be verified that for each $n \geq 1$,

$$\| I_n\big(f_n(\cdot, t, x)\big) \|^2 = \frac{1}{(n!)^2} \mathbb{E}_0 \Big[\int_0^t \int_0^t \gamma_0(s-r) \gamma\big(B(s) - \widetilde{B}(r)\big) ds dr \Big]^n$$

where B and $\widetilde{B}$ are two independent d-dimensional Brownian motions starting at 0.

Whenever possible, the Brownian Hamiltonian

$$\int_0^t \int_0^t \gamma_0(s-r) \gamma\big(B(s) - \widetilde{B}(r)\big) ds dr$$

is defined by approximation. The finiteness of its moment allows us to carry the approximation out.

Relevance to Brownian Hamiltonian

By Taylor expansion, therefore, the solvability is equivalent to the exponential integrability

$$\mathbb{E}\, u^2(t,x) = \mathbb{E}_0 \exp\left\{ \int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr \right\} < \infty$$

Major challenge

The main step is to get the bound

$$\mathbb{E}_0\Big[\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr\Big]^n \leq (n!)^\theta C^n t^{n(2H_0+H-d)}$$

$n=1,2,\cdots$, with $\theta<1$.

When $H_0=1/2$ ($\gamma_0(\cdot)=\delta_0(\cdot)$),

$$\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr = \int_0^t \gamma\big(B(s)-\widetilde{B}(s)\big)ds$$

$$=\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t e^{i\xi\cdot\left(B(s)-\widetilde{B}(r)\right)}ds \stackrel{d}{=} \int_{\mathbb{R}^d}\mu(d\xi)\int_0^t e^{i\sqrt{2}\xi\cdot B(s)}ds$$

Major challenge

Hence,

$$\mathbb{E}_0\left[\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr\right]^n$$

$$=\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\sqrt{2}\xi_k\cdot B(s_k)}\right)ds$$

$$=n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n_<}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\sqrt{2}\xi_k\cdot B(s_k)}\right)ds$$

where

$$[0,t]^n_< = \{(s_1,\cdots,s_n)\in[0,t]^n;\ \ s_1<\cdots<s_n\}$$

and we adopt the simplified notations

$$\mu(d\xi)=\mu(d\xi_1)\cdots\mu(d\xi_n)\ \text{ and }\ ds=ds_1\cdots ds_n$$

in the context whenever it becomes obvious.

Major challenge

With the price $n!$ for ordering $s_1 < \cdots < s_n$, we have the clear evaluation

$$\mathbb{E}_0 \prod_{k=1}^{n} e^{i\sqrt{2}\xi_k \cdot B(s_k)} = \mathbb{E}_0 \exp\left\{ i\sqrt{2} \sum_{k=1}^{n} \Big(\sum_{j=k}^{n} \xi_j \Big) (B(s_k) - B(s_{k-1})) \right\}$$

$$= \exp\left\{ -\sum_{k=1}^{n} \Big| \sum_{j=k}^{n} \xi_j \Big|^2 (s_k - s_{k-1}) \right\}$$

which leads to a sharp bound for the n-moment.

Major challenge

When $H_0 \neq 1/2$, we have a formal moment representation

$$\mathbb{E}_0\left[\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr\right]^n$$
$$= \int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^{2n}}\left(\prod_{k=1}^n \gamma_0(s_k-r_k)\right)$$
$$\times\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)\left(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)}\right)dsdr$$

Should the price $(n!)^2$ be paid for the expectations on the right hand side to be evaluated or bounded?

Major challenge

The proposal of $(n!)^2$-payment is unjustified: To a degree, the mass concentrates on the diagonal $\{s = r\}$. Consequently, re-arranging $\{s_1, \cdots, s_n\}$ should lead to, partially at least, to the same order of $(r_1, \cdots, r_n)$. Hence, the $(n!)^2$-payment would un-necessarily increase the power on $n!$. In $H_0 < 1/2$, it rule out any possibility for the requested exponetial integrability.

Major challenge

Alternative treament is to use the bound

$$0 < \mathbb{E}_0 \prod_{k=1}^{n} e^{i\xi_k \cdot B(r_k)} \leq 1$$

As $H_0 > 1/2$, $\gamma_0(\cdot) \geq 0$, so we have

$$\mathbb{E}_0 \Big[\int_0^t \int_0^t \gamma_0(s-r)\gamma\big(B(s) - \widetilde{B}(r)\big) ds dr \Big]^n$$

$$\leq \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,t]^n} \left(\prod_{k=1}^{n} \int_0^t \gamma_0(s_k - r) dr \right) \left(\mathbb{E}_0 \prod_{k=1}^{n} e^{i\xi_k \cdot B(s_k)} \right) ds$$

which lower the cost on $n!$, but weaken the integrability (to a catastrophic level sometimes).

Major challenge

This practice is not allowed at all when $H < 1/2$ as $\gamma_0(\cdot)$ is sign-switching.

In [Chen, AIHP (to appear)], the the challenge is responded in $H_0 > 1/2$ with a better option

$$\mathbb{E}_0\left[\int_0^1\int_0^1 \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)\,dsdr\right]^n$$
$$\leq C^n\int_{(\mathbb{R}^d)^n}\mu(d\xi)\left[\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\right]^\beta$$

with $\beta \approx 2H_0$. Then the time permutation is performed on the right hand with the total cost of roughly $(n!)^{2H_0}$.

Major challenge

One can prove that this bound does not hold in $H_0 < 1/2$. In [Chen, AIHP (to appear)], we get the bound

$$\mathbb{E}_0\left[\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr\right]^n \le (n!)^\theta C^n t^{n(2H_0+H-d)}$$

with $(n!)^2$-payment strategy. Sadly, $\theta > 1$ in the above bound.

Solving the equation: $H_0 < 1/2$

To improve the bound, we first establish the following decomposition for $H_0 < 1/2$ under the assumptions in our theorem

Lemma

$$\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr$$

$$= H_0\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t \left\{s^{-(1-2H_0)}+(t-s)^{-(1-2H_0)}\right\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds$$

$$+\frac{H_0(1-2H_0)}{2}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t$$

$$\times\frac{\left[e^{i\xi\cdot B_s}-e^{i\xi\cdot B_r}\right]\left[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\right]}{|s-r|^{2-2H_0}}drds$$

Remarks on this lemma

1. This lemma is partially inspired by a deterministic covariance decomposition in Chen, L., Hu, Y. Z., Kalbasi, K. and Nualart, D (PTRF, to appear)

2. The first term is in 1-multiple integral whose n-moment can be well bounded by the $n!$-payment plan.

3. As for the second term, by symmetry it is equal to

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t \frac{\left[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}\right]\left[e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}\right]}{|s-r|^{2-2H_0}} drds$$

$$= 2\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t \frac{e^{i\xi\cdot B_s}\left[e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}\right]}{|s-r|^{2-2H_0}} drds$$

Solving the equation: $H_0 < 1/2$

Proof. By a simple algebra

$$
\begin{aligned}
&\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\widetilde{B}(r)\big)dsdr \\
&= \int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot B_s}e^{-i\xi\cdot\widetilde{B}_r}drds \\
&= \int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot B_s}e^{-i\xi\cdot\widetilde{B}_s}drds \\
&- \int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot B_s}\Big[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\Big]drds
\end{aligned}
$$

The first term on the right hand side is identified with the first term in the decomposition.

Solving the equation: $H_0 < 1/2$

As for the second term, for any $N > 0$

$$\int_{[-N,N]\times\mathbb{R}^d} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)} e^{i\xi\cdot B_s}\left[e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}\right] drds$$

$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \left(\int_{-N}^N e^{i\lambda(s-r)}|\lambda|^{1-2H_0} d\lambda\right) e^{i\xi\cdot B_s}\left[e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}\right] drds$$

$$= N^{1-2H_0}\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \frac{\sin N(s-r)}{s-r} e^{i\xi\cdot B_s}\left[e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}\right] drds$$

$$- \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \left(\int_{-N}^N \frac{\sin\left(\lambda(s-r)\right)}{(s-r)|\lambda|^{2H_0}} d\lambda\right) e^{i\xi\cdot B_s}\left[e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}\right] drds$$

Our claim follows from the fact that the first term goes to zero as $N \to \infty$.

Solving the equation: $H_0 < 1/2$

To prove our main theorem, all we need is to establish a good n-moment bound for the second term in the decomposition, which is the constant multiple of

$$Q_t = \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t \frac{e^{i\xi\cdot B_s}\left[e^{-i\xi\cdot \widetilde{B}_s} - e^{-\xi\cdot \widetilde{B}_r}\right]}{|s-r|^{2-2H_0}} drds$$

By assumption there is a $\beta > 0$ such that

$$1 - 2H_0 < \beta < \frac{1}{2} - \frac{2(d-H) + (d_* - 2H_*)}{4}$$

All we need is the bound

$$\mathbb{E}_0 Q_t^n \le (n!)^{(d-H)+2\beta} C^n t^{n(2H_0+H-d)}$$

as $(d-H) + 2\beta < 1$.

Solving the equation: $H_0 < 1/2$

We have $Q_t \overset{d}{=} t^{2H_0+H-d} Q_1$. We may let $t = 1$, i.e.,

$$\mathbb{E}_0 Q_1^n \le (n!)^{(d-H)+(1-2H_0)} C^n$$

We now start the moment computation. Notice that

$$Q_1 = 2\int_{\mathbb{R}^d} \mu(d\xi) \int_0^1 \int_0^1 \frac{e^{i\xi\cdot(x+B_s)} e^{-i\xi\cdot(\tilde{x}+\widetilde{B}_s)} \sin^2 \frac{\xi\cdot(\widetilde{B}_s-\widetilde{B}_r)}{2}}{|s-r|^{2-2H_0}} drds$$

Solving the equation: $H_0 < 1/2$

$$
\begin{aligned}
\mathbb{E}_0 Q_1^n &= 2^n \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k \cdot B_{s_k}} \right) \\
&\times \left\{ \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{-i\xi \cdot B_{s_k}} |s_k - r_k|^{-(2-2H_0)} \sin^2 \frac{\xi_k \cdot (B_{s_k} - B_{r_k})}{2} \right) dr \right\} ds \\
&\leq 2^n \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k \cdot B_{s_k}} \right) \\
&\times \left\{ \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)} \sin^2 \frac{\xi_k \cdot (B_{s_k} - B_{r_k})}{2} \right) dr \right\} ds
\end{aligned}
$$

Solving the equation: $H_0 < 1/2$

Picking $1 - 2H_0 < \beta_1 < \beta$

$$\int_{[0,1]^n} \left(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)} \right) \left(\mathbb{E}_0 \prod_{k=1}^n \sin^2 \frac{\xi_k \cdot (B_{s_k} - B_{r_k})}{2} \right) dr$$

$$\leq \int_{[0,1]^n} \left(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)} \right) \left(\mathbb{E}_0 \prod_{k=1}^n \left| \sin \frac{\xi_k \cdot (B_{s_k} - B_{r_k})}{2} \right|^{2\beta} \right) dr$$

$$\leq \left(\prod_{k=1}^n |\xi_k|^{2\beta} \right) \mathbb{E}_0 \left(\sup_{0 \leq r, s \leq 1} \frac{|B_s - B_r|}{|s - r|^{\beta_1/(2\beta)}} \right)^{2\beta n}$$

$$\times \int_{[0,1]^n} \left(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0-\beta_1)} \right) dr$$

Solving the equation: $H_0 < 1/2$

By the fact that $2 - 2H_0 - \beta_1 < 1$,

$$\int_{[0,1]^n} \left(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0-\beta_1)} \right) dr \le C^n$$

So we have the bound

$$\mathbb{E}_0 Q_1^n \le C^n \mathbb{E}_0 \left(\sup_{0 \le r,s \le 1} \frac{|B_s - B_r|}{|s-r|^{\beta_1/(2\beta)}} \right)^{2\beta n}$$
$$\times \int_{(\mathbb{R}^d)^n} \mu(d\xi) \left(\prod_{k=1}^n |\xi_k|^{2\beta} \right) \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k \cdot B_{s_k}} \right) ds$$

Solving the equation: $H_0 < 1/2$

Notice $0 < \beta_1/(2\beta) < 1/2$. By the Hölder continuity of the Brownian motion and by Gaussian tail bound

$$\mathbb{E}_0\left(\sup_{0\le r,s\le 1}\frac{|B_s - B_r|}{|s-r|^{\beta_1/(2\beta)}}\right)^{2\beta n} \le (n!)^\beta C^n$$

Therefore,

$$E_0 Q_1^n \le (n!)^\beta C^n \int_{(\mathbb{R}^d)^n}\mu(d\xi)\left(\prod_{k=1}^n|\xi_k|^{2\beta}\right)\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\right)ds$$

It remains to prove

$$\int_{(\mathbb{R}^d)^n}\mu(d\xi)\left(\prod_{k=1}^n|\xi_k|^{2\beta}\right)\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\right)ds$$
$$\le (n!)^{(d-H)+\beta}C^n$$

Solving the equation: $H_0 < 1/2$

Write

$$I_n(t) = \int_{(\mathbb{R}^d)^n} \mu(d\xi)\left(\prod_{k=1}^n |\xi_k|^{2\beta}\right)\int_{[0,t]^n}\left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\right)ds$$

Then $I_n(t) = t^{n(1-(d-H)-\beta)} I_n(1)$ and

$$I_n(t) = n!\int_{(\mathbb{R}^d)^n} \mu(d\xi)\left(\prod_{k=1}^n |\xi_k|^{2\beta}\right)\int_{[0,t]^n_<}\left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\right)ds$$

$$= n!\int_{(\mathbb{R}^d)^n} \mu(d\xi)\left(\prod_{k=1}^n |\xi_k|^{2\beta}\right)$$

$$\times\int_{[0,t]^n_<} \exp\left\{-\frac{1}{2}\sum_{k=1}^n\left|\sum_{j=k}^n \xi_j\right|^2 (s_k - s_{k-1})\right\}ds$$

Solving the equation: $H_0 < 1/2$

Hence,

$$\int_0^\infty e^{-t} I_n(t)dt = n! \int_{(\mathbb{R}^d)^n} \mu(d\xi)\Big(\prod_{k=1}^n |\xi_k|^{2\beta}\Big)$$
$$\times \prod_{k=1}^n \int_0^\infty e^{-t} \exp\Big\{-\frac{1}{2}\Big|\sum_{j=k}^n \xi_j\Big|^2 t\Big\}dt$$
$$= n! \int_{(\mathbb{R}^d)^n} \mu(d\xi)\Big(\prod_{k=1}^n |\xi_k|^{2\beta}\Big) \prod_{k=1}^n \Big\{1 + \frac{1}{2}\Big|\sum_{j=k}^n \xi_j\Big|^2\Big\}^{-1}$$

Solving the equation: $H_0 < 1/2$

Notice $0 < 2\beta < 1$ and write $\eta_n = \sum_{j=k}^{n} \xi_j$ $(1 \le j \le n)$. Under the convention $\eta_{n+1} = 0$

$$\prod_{k=1}^{n} |\xi_k|^{2\beta} = \prod_{k=1}^{n} |\eta_k - \eta_{k+1}|^{2\beta} \le \prod_{k=1}^{n} \left\{ |\eta_k|^{2\beta} + |\eta_{k+1}|^{2\beta} \right\}$$

$$\le \sum_{l} \prod_{k=1}^{n} |\eta_k|^{2l(k)\beta} \le 2^n \sum_{l} \prod_{k=1}^{n} \left\{ 1 + \frac{1}{2} |\eta_k|^2 \right\}^{l(k)\beta}$$

$$\le 2^n \sum_{l} \prod_{k=1}^{n} \left\{ 1 + \frac{1}{2} |\eta_k|^2 \right\}^{2\beta} \le 2^n 3^n \prod_{k=1}^{n} \left\{ 1 + \frac{1}{2} \Big| \sum_{j=k}^{n} \xi_j \Big|^2 \right\}^{2\beta}$$

where the summation is over all maps l: $\{1, \cdots, n\} \longrightarrow \{0, 1, 2\}$ so the number of its terms is at most 3^n.

Solving the equation: $H_0 < 1/2$

Summarizing our computation,

$$\int_0^\infty e^{-t} I_n(t)dt \le n! C^n \int_{(\mathbb{R}^d)^n} \mu(d\xi) \prod_{k=1}^n \left\{ 1 + \frac{1}{2} \Big| \sum_{j=k}^n \xi_j \Big|^2 \right\}^{-(1-2\beta)}$$
$$\le n! C^n$$

where the last step follows from the fact that

$$1 - 2\beta > \frac{2(d - H) + (d_* - 2H_*)}{2}$$

and the lamma stated later.

Lemmas on integrability

On the other hand,

$$\int_0^\infty e^{-t} I_n(t)dt = I_n(1)\int_0^\infty e^{-t} t^{n(1-(d-H)-\beta)}dt$$
$$= I_n(1)\Gamma\big(1 + n(1-(d-H)-\beta)\big)$$

By Stirling formula,

$$I_n(1) \le (n!)^{(d-H)+\beta} C^n$$

□

Further development

One of important properties in SPDE is the intermittency. It described by the asymptotic behavior

$$\log \mathbb{E}\, u^m(t,x) \quad (t \to \infty) \quad m = 2, 3, \cdots$$

For $H_0 > 1/2$, the answer (Chen, AIHP (to appear)) to this question is

$$\lim_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}\, u^m(t,x) = \left(\frac{1}{2}\right)^{\frac{1}{1-(d-H)}} m(m-1)^{\frac{1}{1-(d-H)}} \mathcal{E}(\mathbf{H})$$

where $\mathcal{E}(\mathbf{H}) > 0$ is a constant given in terms of variation.

Further development

Notice that $\frac{2H_0+H-d}{1-(d-H)} < 1$ as $H_0 < 1/2$. Does the logarithmic moment

$$\log \mathbb{E}\, u^m(t,x)$$

has a sub-linear growth rate when $H_0 < 1/2$ as $t \to \infty$? The truth is that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\, u^2(t,x) = \kappa(\mathbf{H})$$

for some constant $0 < \kappa(\mathbf{H}) < \infty$.

More problems need to be answered in the future on the intermittency in the setting of $H_0 < 1/2$.

Lemmas on integrability

Lemma

$$\int_{(\mathbb{R}^d)^n} \mu(d\xi) \prod_{k=1}^{n} \left\{ 1 + \frac{1}{2} \Big| \sum_{j=k}^{n} \xi_j \Big|^2 \right\}^{-\kappa} \le C^n$$

for any

$$\kappa > \frac{2(d - H) + (d_* - 2H_*)}{2}$$

To this end, we first prove

Lemmas on integrability

Lemma

Let $f(\xi)$ and $g(\xi)$ be two non-negative definite functions on $\mathbb{R}^d$. Then for any $\xi \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} f(\eta)g(\eta-\xi)d\eta \leq \int_{\mathbb{R}^d} f(\eta)g(\eta)d\eta$$

Proof. Let $\mu_f(dx)$ and $\mu_g(dx)$ be the spectral measures of f and g, respectively. Assume $\mu_f(dx) = \hat{f}(x)dx$ and $\mu_g(dx) = \hat{g}(x)dx$ for some $\hat{f}, \hat{g} \geq 0$.

$$\int_{\mathbb{R}^d} f(\eta)g(\eta-\xi)d\eta = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\hat{f}(x)\hat{g}(x)dx$$
$$\leq \int_{\mathbb{R}^d} \hat{f}(x)\hat{g}(x)dx = \int_{\mathbb{R}^d} f(\eta)g(\eta)d\eta$$

□

Lemmas on integrability

We now prove the bound

$$\int_{(\mathbb{R}^d)^n} \mu(d\xi) \prod_{k=1}^{n} \left\{ 1 + \frac{1}{2} \Big| \sum_{j=k}^{n} \xi_j \Big|^2 \right\}^{-\kappa} \le C^n$$

We may assume that $\kappa \le 2$ in the following proof.

Recall that $J^* = \{1 \le j \le d;\ H_j \ge 1/2\}$ and $J_* = \{1 \le j \le d;\ H_j < 1/2\}$.

Lemma on integrability

In the notation $\xi_k = (\xi_{k,1}, \cdots, \xi_{k,d})$, $\xi_k^+ = (\xi_{k,j})_{j\in J^*}$ and $\xi_k^- = (\xi_{k,j})_{j\in J_*}$

$$\begin{aligned}\mu(d\xi) &= C^n \prod_{k=1}^n \left(\prod_{j=1}^d |\xi_{k,j}|^{1-2H_j}\right) d\xi_k \\ &= C^n \prod_{k=1}^n \left(\prod_{j\in J^*} |\xi_{k,j}|^{1-2H_j}\right)\left(\prod_{j\in J_*} |\xi_{k,j}|^{1-2H_j}\right) d\xi_k \\ &= C^n \prod_{k=1}^n q^*(\xi_k^+) q_*(\xi_k^-) d\xi_k^+ d\xi_k^- \quad \text{(say)}\end{aligned}$$

Lemmas on integrability

By translation,

$$\int_{(\mathbb{R}^d)^n} \prod_{k=1}^{n} \left(1 + \frac{1}{2}\Big|\sum_{k=j}^{n} \xi_k\Big|^2\right)^{-\kappa} \mu(d\xi)$$

$$= C^n \int_{(\mathbb{R}^{J_*} \times \mathbb{R}^{J^*})^n} \left\{ \prod_{k=1}^{n} \left(1 + \frac{1}{2}|\xi_k^-|^2 + \frac{1}{2}|\xi_k^+|^2\right)^{-\kappa} \right\}$$

$$\times \prod_{k=1}^{n} q^*(\xi_k^+ - \xi_{k-1}^+) q_*(\xi_k^- - \xi_{k-1}^-) d\xi_k^+ d\xi_k^-$$

$$\leq C^n \int_{(\mathbb{R}^{J_*} \times \mathbb{R}^{J^*})^n} \left\{ \prod_{k=1}^{n} \left((1 + |\xi_k^-|^2)^{\kappa/2} + |\xi_k^+|^\kappa\right)^{-2} \right\}$$

$$\times \prod_{k=1}^{n} q^*(\xi_k^+ - \xi_{k-1}^+) q_*(\xi_k^- - \xi_{k-1}^-) d\xi_k^+ d\xi_k^-$$

Lemmas on integrability

Notice that the function $q^*(\eta)$ $(\eta \in \mathbb{R}^{J^*})$ is non-negative definite with spectral density $\hat{q}^*(x)$ which appears to be the constant multiple of

$$\prod_{j\in J^*} |x_j|^{-(2-2H_j)} \qquad x = (x_j)_{j\in J^*} \in \mathbb{R}^{J^*}$$

Also notice that for any $a > 0$, $f(\eta) = (a + |\eta|^\kappa)^{-1}$ $(\eta \in \mathbb{R}^{J^*})$ is non-negative definite, which appears to the characteristic function of a κ-stable and radius-symmetric process at a independent exponential time. Consequently, the function $(a + |\eta|^\kappa)^{-2}$ is non-negative definite.

Lemmas on integrability

By the previous lemma, for any $\zeta \in \mathbb{R}^{J^*}$

$$\int_{\mathbb{R}^{J^*}} (a + |\eta|^\kappa)^{-2} q^*(\eta - \zeta) d\eta \leq \int_{\mathbb{R}^{J^*}} (a + |\eta|^\kappa)^{-2} q^*(\eta) d\eta$$
$$= a^{-2+2\kappa^{-1}(d^* - H^*)} \int_{\mathbb{R}^{J^*}} (1 + |\eta|^\kappa)^{-2} q^*(\eta) d\eta$$

This implies that for any $a_1, \cdots, a_n > 0$,

$$\int_{(\mathbb{R}^{J^*})^n} \left(\prod_{k=1}^n (a_k + |\xi_k^+|^\kappa)^{-2} \right) \prod_{k=1}^n q^*(\xi_k^+ - \xi_{k-1}^+) d\xi_k^+$$
$$\leq C^n \prod_{k=1}^n a_k^{-2+2\kappa^{-1}(d^* - H^*)}$$

Lemmas on integrability

Take

$$a_k = (1 + |\xi_k^-|^2)^{\kappa/2}$$

By Fubini's theorem,

$$\int_{(\mathbb{R}^d)^n} \prod_{k=1}^{n} \left(1 + \frac{1}{2}\Big|\sum_{k=j}^{n} \xi_k\Big|^2\right)^{-\kappa} \mu(d\xi)$$
$$\leq C^n \int_{(\mathbb{R}^{J_*})^n} \left(\prod_{k=1}^{n} (1 + |\xi_k|^2)^{-\kappa + (d^* - H^*)}\right) \prod_{k=1}^{n} q_*(\xi_k - \xi_{k-1}) d\xi_k$$

Here we use ξ_k instead of ξ_k^- on the right hand side for notation simplification.

Lemmas on integrability

Notice that

$$\prod_{k=1}^{n} q_*(\xi_k - \xi_{k-1}) = \prod_{k=1}^{n}\prod_{j\in J_*} |\xi_{k,j} - \xi_{k-1,j}|^{1-2H_j}$$

$$\leq \prod_{k=1}^{n}\prod_{j\in J_*}\left(|\xi_{k,j}|^{1-2H_j} + |\xi_{k-1,j}|^{1-2H_j}\right)$$

$$\leq \sum_{l}\prod_{k=1}^{n}\prod_{j\in J_*} |\xi_{k,j}|^{l(k,j)(1-2H_j)}$$

where the summation is taken for all maps l: $\{1,\cdots,n\}\times J_* \longrightarrow \{0,1,2\}^n$ with

$$\sum_{(k,j)\in\{1,\cdots,n\}\times J_*} l(k,j) = n$$

and therefore the number of the terms is at most 2^{nd_*}.

Lemmas on integrability

Therefore, all we need to prove is that

$$\int_{\mathbb{R}^{J_*}} \left(1 + |\xi|^2\right)^{-\kappa+(d^*-H^*)} \prod_{j\in J_*} |\xi_j|^{l(1-2H_j)} d\xi < \infty \qquad l = 0, 1, 2$$

Notice that $1 - 2H_j > 0$ for each $j \in J_*$. So only the case $l = 2$ needs to be checked. Indeed, by spherical substitution

$$\int_{\mathbb{R}^{J_*}} \left(1 + |\xi|^2\right)^{-\kappa+(d^*-H^*)} \prod_{j\in J_*} |\xi_j|^{2(1-2H_j)} d\xi$$
$$= C \int_0^\infty \left(1 + \rho^2\right)^{-\kappa+(d^*-H^*)} \rho^{2(d_*-2H_*)} \rho^{d_*-1} d\rho < \infty$$

□

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Thank you!